2. (Quasi)Affine algebraic varieties

Def: An affine algebraic variety V is an irreducible affine algebraic set.

2.1 Zariski topology

Def: The Zariski topology on $\mathbb{A}^n$ is the topology whose closed sets are algebraic set. The open sets are complements of closed sets.

Lemma 2.1: This actually defines a topology on $\mathbb{A}^n$.

pf: $\emptyset, \mathbb{A}^n$ are algebraic $\rightsquigarrow \mathbb{A}^n, \emptyset$ are open.

Let $\{U_i\}_{i\in I}$ be a family of open sets i.e. $U_i = \mathbb{A}^n \setminus V(I_i)$

$\rightsquigarrow \bigcup_{i\in I} U_i = \mathbb{A}^n \setminus \bigcap V(I_i) = \mathbb{A}^n \setminus V(\bigcup I_i)$ is again open.

U_1, U_2 open. Then $U_1 \cap U_2 = \mathbb{A}^n \setminus \{V(I_1) \cup V(I_2)\} = \mathbb{A}^n \setminus V(I_1 I_2)$

$\square$

Exple: The only closed sets in $\mathbb{A}^1$ are $\mathbb{A}^1, \emptyset$ and finite sets of points i.e. Zariski = cofinite topology.
i.e. Open sets are Huge

Def: Let $V \subset \mathbb{A}^n$ be an affine alg. var. The Zariski topology on V is the subspace topology induced by the Zariski topo. on $\mathbb{A}^n$.

Rmk: At first it might seem strange to use a topology that differs so much from our intuition, but it turns out to be extemly usefull:

(New) Def: A non-empty subset V of a topological space X is irreducible if it cannot be expressed as $V = W_1 \cup W_2$, where $W_1, W_2 \subsetneq V$ closed in V.

This clearly generalizes irreducibility as we defined it before.

Lemma 2.2: · A non-empty open subset of an irreducible space is irreducible and dense.

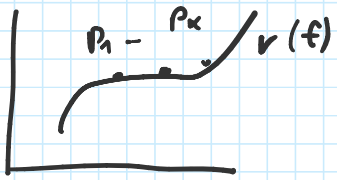
· If $V \subset X$ is irred., then so is $\bar{V}$.

pf: Exercise

· A quasi-affine variety is an open subset of an affine variety.

By the previous lemma quasi-affine var. are irred.

Exple: · $\mathbb{A}^2 \setminus \{0\}$, $\mathbb{A}^2 \setminus V(f)$, $f \in K[x,y]$.



$V(f) \setminus \{p_1, .., p_k\}$.

2.2. Regular functions and coordinate rings

2.2.1. Affine case

Regular functions are the natural continous functions on quasi-affine varieties. Let's start with the affine case:

Def: Let $V \subset \mathbb{A}^n$ be an affine variety. A map

$$f : V \to \mathbb{A}^1 = K$$

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is **regular** if $\exists\, F \in k[X_1, \dots, X_n]$ s.t.

$$f(x) = F(x) \qquad \forall x \in V \subset \mathbb{A}^n.$$

The set $\Gamma(V)$ of regular functions on V is a ring with the usual definition of addition and multiplication. It is called the coordinate ring of V

Lemma 2.3: If $V = V(I)$ for a prime ideal $I \subset k[X_1, \dots, X_n]$ then $\Gamma(V) \simeq k[X_1, \dots, X_n]/_I$. I.p. $\Gamma(V)$ is a domain

pf: By definition we have a surj ring homom.

$$\varphi : k[X_1, \dots, X_n] \to \Gamma(V)$$
$$F \mapsto F_{|V}$$
, but $F \in \ker(\varphi) \iff F_{|V} \equiv 0$
$\iff F \in I$ $\square$

Def: A affine subvariety W of V is an affine variety contained in V.

Lemma 2.4: There is a 1-1 correspondence between

V	$\Gamma(V)$
alg subsets	radical ideals
affine subvar	prime ideals
points	maximal ideals

pf: Exercise

Rmk: We'll often write m_p or $m_p(V)$ for the maximal ideal corresp. to a point $p \in V$.

Def: A morphism $\varphi : V \to W$ between

<u>Def</u>: A <u>morphism</u> $\varphi : V \to W$ between affine alg. var. $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$ is a map s.t. $\exists$ polynomials $T_1, \dots, T_m \in k[x_1, \dots, x_n]$ s.t.

$$\varphi(x) = (T_1(x), \dots, T_m(x)).$$

- φ is an <u>isomorphism</u> if $\exists$ a morphism $\psi : W \to V$ s.t. $\psi \circ \varphi = 1_V$, $\varphi \circ \psi = 1_W$.

<u>Exple</u>: • $V = V(x^2 - y) \subset \mathbb{A}^2$

proj the x coord.

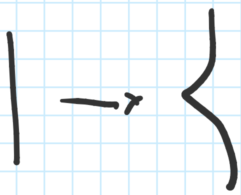
Then $V \simeq \mathbb{A}^1$

$(x,y) \mapsto x$ since $y = x^2$ on V, this is an isom.

$(x, x^2) \leftarrow\!\shortmid\ x$

• $\varphi : \mathbb{A}^1 \to V(y^2 - x^3) \subset \mathbb{A}^2$

$t \mapsto (t^2, t^3)$



One can check that φ is a bijective morphism but not an isomorphism! (Exercise)

In general any morphism $\varphi : V \to W$ induces a ring homom. $\tilde{\varphi} : \Gamma(W) \to \Gamma(V)$ by composition

$$\tilde{\varphi}(f) := f \circ \varphi$$

Clearly if $V \xrightarrow{\varphi} W \xrightarrow{\psi} Z$, $\widetilde{\psi \circ \varphi} = \tilde{\varphi} \circ \tilde{\psi}$

<u>Proposition 2.5</u>: This defines a 1-1 corresp.

$$\left\{ \begin{array}{l} \text{Morphisms} \\ \varphi : V \to W \end{array} \right\} \underset{\varphi \mapsto \tilde{\varphi}}{\longrightarrow} \left\{ \begin{array}{l} k\text{-alg homom.} \\ \alpha : \Gamma(W) \to \Gamma(V) \end{array} \right\},$$

In particular φ is an iso. $\iff$ $\tilde{\varphi}$ is an iso.

pf: Need to construct for any $\alpha : \Gamma(W) \to \Gamma(V)$ a morph. $\bar{\alpha} : V \to W$ s.t. $\tilde{\bar{\alpha}} = \alpha$ and $\bar{\tilde{\varphi}} = \varphi$.

Supp. $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$ and write $\Gamma(V) = K[X_1, \dots, X_n]/I(V)$, $\Gamma(W) = K[Y_1, \dots, Y_m]/I(W)$.

Choose lifts T_i of $\alpha([Y_i])$ in $K[X_1, \dots, X_n]$ for $1 \le i \le m$.

We have a comm. diagram $Y_i \mapsto T_i$

$$\begin{array}{ccc} K[Y_1 \dots Y_m] & \longrightarrow & K[X_1, \dots, X_n] \\ \downarrow & & \downarrow \\ \Gamma(W) & \xrightarrow{\alpha} & \Gamma(V) \end{array}$$

In part. $\forall f \in \Gamma(W)$ and $F \in K[Y_1, \dots, Y_m]$ a lift, we have

$$\alpha(f) \equiv F(T_1, \dots, T_m) \mod I(V).$$

Let $T : \mathbb{A}^n \to \mathbb{A}^m$ be given by $T = (T_1, \dots, T_m)$

Claim: $T(V) \subset W$

pf: From the diagram we see $\forall G \in I(W)$

$$G(T_1, \dots, T_m) \in I(V)$$

$\leadsto \forall v \in V \quad 0 = G(T_1, \dots, T_m)(v) = G(T(v)) \leadsto T(v) \in W$ claim $\square$

We set $\bar{\alpha} := T_{|V} : V \to W$.

Now $\tilde{\bar{\alpha}} : \Gamma(W) \to \Gamma(V)$ satisfies $\forall v \in V \ \forall f \in \Gamma(W)$

$$\tilde{\bar{\alpha}}(f)(v) = f(\bar{\alpha}(v)) = f(T_1(v), \dots, T_m(v)) = \alpha(f)(v)$$

$\leadsto \tilde{\bar{\alpha}} = \alpha$. The pf of $\bar{\tilde{\alpha}} = \alpha$ is similar $\square$

Def: The quotient field $k(V)$ of $\Gamma(V)$ is called <u>the field of rational functions on V</u>.

- $f \in k(V)$ <u>is defined</u> at a point $p \in V$ if we can write $f = \frac{a}{b}$ w $a, b \in \Gamma(V)$ and $b(p) \neq 0$.
- The <u>pole set</u> of $f \in k(V)$ is the set of points where f is not defined

Rmk: $\Gamma(V)$ is not a UFD in general, so the presentation $f = \frac{a}{b}$ is not unique in any sense. For example $V = V(xy - zw) \subset \mathbb{A}^4$, let $\bar{X}, \bar{Y}, \bar{Z}, \bar{W}$ be the image of X, Y, Z, W in

$$\Gamma(V) = k[X, Y, Z, W] / XY - ZW.$$

Then $f = \frac{\bar{X}}{\bar{W}} = \frac{\bar{Z}}{\bar{Y}} \in k(V)$. Hence f is defined at $P = (x, y, z, w) \in \Gamma(V)$ $\Leftrightarrow$ $y \neq 0$ or $w \neq 0$.

Def: The <u>local ring of V at $p \in V$</u> is the subring of $k(V)$ defined by $\mathcal{O}_p(V) := \{ f \in k(V) \mid f \text{ is defined at } p \}$.

We have natural inclusions $\Gamma(V) \subset \mathcal{O}_p(V) \subset k(V)$.

Rmk: $\Gamma(V)$, $\mathcal{O}_p(V)$ and $k(V)$ are intrinsic to V by Prop. 2.5

Since if $V \xrightarrow[\sim]{\varphi} W \Rightarrow \Gamma(V) \simeq \Gamma(W) \Rightarrow k(V) \simeq k(W)$
$\Rightarrow \mathcal{O}_p(V) \simeq \mathcal{O}_{\varphi(p)}(W)$.

Proposition 2.6: Let $p \in V$ and $m_p \subset \Gamma(V)$ the corresp. maximal ideal. Then $\mathcal{O}_p(V)$ is the localization of $\Gamma(V)$ at m_p. In part. $\mathcal{O}_p(V)$ is a Noetherian local domain and we have

$$\Gamma(V) = \bigcap_{p \in V} \mathcal{O}_p(V) \subset k(V)$$

Pf: Recall that $m_p = \{ f \in \Gamma(V) \mid f(p) = 0 \}$. Since $\Gamma(V)$ is a domain $\quad \Gamma(V)_{m_p} = \{ f \in K(V) \mid f = \frac{a}{b} \text{ w/ } b \notin m_p \}$

$$= \{ \quad '' \quad \mid \quad '' \quad b(p) \neq 0 \} = \mathcal{O}_p(V)$$

The rest follows from standart properties of localization, i.p. for any domain R we have

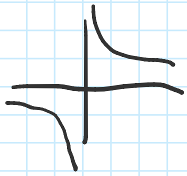
$$R = \bigcap_{m \subset R \text{ max.}} R_m \qquad \square$$

__Rmk:__ One should think of $\mathcal{O}_p(V)$ as functions in a nbhd of $p \in V$.

Notice that the notion of regular functions is sufficient to define morphism, local rings, rational fct's.

How can we extend this to quasi-affine varieties?

__Expl:__ $V(xy-1) \subset \mathbb{A}^2 \xrightarrow{\varphi} \mathbb{A}^1$. The image of φ is the quasi-
$(x,y) \mapsto x$



affine var. $\mathbb{A}^1 \setminus \{0\}$, and we'd like φ to be an iso. But the inverse of φ is $x \mapsto (x, \frac{1}{x})$.

$\rightsquigarrow$ $x \mapsto \frac{1}{x}$ should be a regular map on $\mathbb{A}^1 \setminus \{0\}$.

2.2.2. Quasi-affine case:

__Def:__ Let V be a quasi-aff. var. contained in $\mathbb{A}^n$. A map $\quad f: V \to k$ is __regular__, if for every point $v \in V$ $\exists$ an open nbhd $v \in U \subset V$ and $g, h \in k[x_1, \dots, x_n]$ s.t. $h(x) \neq 0 \; \forall x \in U$ and $f(x) = \frac{g(x)}{h(x)} \quad \forall x \in U$.

s.t. $h(x) \neq 0 \ \forall x \in U$ and $f(x) = \frac{g(x)}{h(x)} \ \forall x \in U$.

Let $\mathcal{O}(V)$ be the <u>ring of regular fctn's</u> on V.

<u>Expl</u>: $f: V = \mathbb{A}^1 \setminus \{0\} \longrightarrow k$, $x \longmapsto \frac{1}{x}$ is regular, as we may take $U = V$.

It is not hard to see that $\mathcal{O}(V) = k[x][\frac{1}{x}, \frac{1}{x^2}, \dots] = k[x, \frac{1}{x}] = k[x,y]/_{xy-1}$

i.e. $\mathcal{O}(V) \supsetneq \mathcal{O}(\mathbb{A}^1) = k[x]$.

- If $V = V(XY - ZW) \setminus V(Y, W)$, $\quad f: V \longrightarrow k$, $(x,y,z,w) \longmapsto \begin{cases} \frac{x}{w} & \text{if } w \neq 0 \\ \frac{z}{y} & \text{if } y \neq 0 \end{cases}$

Then there is no global presentation as $f = \frac{g}{h}$ on V

but $f = \frac{\overline{X}}{\overline{W}}$ on $V \setminus (V \cap V(W))$ and

$f = \frac{\overline{Z}}{\overline{Y}}$ on $V \setminus (V \cap V(Y))$. $\rightsquigarrow$ that's why we need U in the definition.

<u>Rmk</u>: If $V \subset \mathbb{A}^n$ is affine we have a natural homom.

$k[X_1, \dots, X_n] \longrightarrow \mathcal{O}(V)$ with kernel $I(V)$
$f \longmapsto f_{|V}$

$\rightsquigarrow \Gamma(V) \hookrightarrow \mathcal{O}(V)$.

<u>Proposition 2.7</u>: For V affine we have $\Gamma(V) \cong \mathcal{O}(V)$.

<u>pf</u>: We have that $\mathcal{O}(V) \subset \mathcal{O}_p(V) = \underbrace{\Gamma(V)_{m_p}}_{\text{prop 2.6.}} \ \forall p \in V$.

$$\Gamma(V) \hookrightarrow \mathcal{O}(V) \subset \bigcap_{p \in V} \mathcal{O}_p(V) \underset{\text{prop. 2.6.}}{=} \Gamma(V)$$

□

<u>Lemma 2.8</u>: V quasi-affine, $f: V \to k$ regular.

Then f is continuous if k is identified with $\mathbb{A}^1$.

<u>pf</u>: Enough to show that f^{-1} of a closed subset is closed.

pf: Enough to show that f^{-1} of a closed subset is closed.
$\Leftrightarrow$ $f^{-1}(a)$ is closed $\forall a \in K$, since the closed subsets of $\mathbb{A}^1$ are $\emptyset, \mathbb{A}^1$ and finite sets of points.

Let $V = \bigcup U_i$ a cover s.t. $f_{|U_i} = \frac{g_i}{h_i}$ and $a \in K$.
Then $f^{-1}(a) \cap U_i = \{v \in U_i \mid g_i(v) = h_i(v) \cdot a\} = V(g_i - h_i \cdot a) \cap U_i$ is closed in U_i $\rightsquigarrow$ $f^{-1}(a)$ closed in V □

Corollary 2.9: $f, g \in \mathcal{O}(V)$ and $U \subset V$ open, non-empty, s.t. $f_{|U} = g_{|U}$. Then $f = g$.
pf: By Lemma 2.2., U is dense in V i.e. $\bar{U} = V$
Since f, g are cont. $f_{|\bar{U}} = g_{|\bar{U}}$ $\rightsquigarrow$ $f = g$ □

Rmk: Let $U \subset V$ be open, then restriction of functions defines an (inj.) map $\mathcal{O}(V) \to \mathcal{O}(U)$
i.e. $\mathcal{O}(\cdot)$ defines a sheaf of rings on V.

Using this, one can define a general alg. var. as a topol. space X w/ a sheaf $\mathcal{O}_X$ which looks "locally" like an affine variety V w/ $\mathcal{O}(\cdot)$. $\rightsquigarrow$ Notion of a scheme (content of Modern alg. geom.)

- We'll define $\mathcal{O}_p(V)$ and $K(V)$ for V quasi-affine varieties later, but since these depend on local str. we can already guess $\mathcal{O}_p(V) = \mathcal{O}_p(\bar{V})$, $K(V) = K(\bar{V})$.